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COMMENT

Infinite conservation laws in a one-dimensional small-polaron model

Huan-Qiang Zhou†‡, Zhuang Xiong§ and Lin-Jie Jiang§

† Center of Theoretical Physics, CCAST (World Laboratory), Beijing, People's Republic of China

§ Institute of Solid State Physics, Sichuan Normal University, Chengdu 610066, People's Republic of China

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Abstract. The local monodromy matrix for a small-polaron model describing the motion of an additional electron in a polaronic crystal is presented by relating it to a special asymmetric six-vertex model. The generating functional for the infinite conservation laws in the model is written down. Moreover, the explicit expression for the first non-trivial conserved current next to the Hamiltonian is constructed.

Recently, Pu and Zhao [1] dealt with the exact solution of a small-polaron model describing the motion of an additional electron in a polar crystal. The approach, however, is not unblemished; in fact, the model Hamiltonian cannot be expressed in terms of the logarithmic derivatives of the transfer matrix constructed from their local monodromy matrix. As is well known in the quantum inverse scattering method (QISM), this is a criterion for whether or not a system is a completely integrable one [2, 3]. As a result of this improper choice for the local monodromy matrix, the energy eigenvalue given by Pu and Zhao is not consistent with that obtained by using the coordinate Bethe-Ansatz technique. In this comment we re-examine this problem by relating it to a special asymmetric six-vertex model. This also allows us to construct explicitly the infinite conservation laws in the model [4-6].

The model Hamiltonian may be written in the form

$$\mathscr{H}=(\varepsilon+W)\sum_{k=1}^{N} n_k - J\sum_{k=1}^{N}(a_k^+a_{k+1}+a_{k+1}^+a_k)+V\sum_{k=1}^{N} n_kn_{k+1}. \tag{1}$$

Here the concrete expressions for W, J and V, along with a discussion of the basic assumptions used in getting (1), can be found in the paper of Fedyanin and Yushankay [7]. Upon using the well known Jordan-Wigner transformation [8, 9] for a_j^+, a_j and n_j

$$a_j^+=\left[\exp\left(\pi\mathrm{i}\sum_{l=1}^{j-1}\sigma_l^+\sigma_l^-\right)\right]\sigma_j^+ \qquad n_j=\tfrac{1}{2}(\sigma_j^z+1)$$
$$a_j=\left[\exp\left(\pi\mathrm{i}\sum_{l=1}^{j-1}\sigma_l^+\sigma_l^-\right)\right]\sigma_j^- \tag{2}$$

‡ Permanent address: Institute of Solid State Physics, Sichuan Normal University, Chengdu 610066, People's Republic of China.

with $\sigma_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm \mathrm{i}\sigma_j^y)$ and σ_j^x, σ_j^y, σ_j^z being Pauli spin matrices at lattice site j, we obtain instead of (1) the Hamiltonian in terms of Pauli spin matrices:

$$H = -\sum_{j=1}^{N} [\tfrac{1}{2}J(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) - \tfrac{1}{4}V\sigma_j^z\sigma_{j+1}^z] + \tfrac{1}{2}(W+\varepsilon+V)\sum_{j=1}^{N}\sigma_j^z + \tfrac{1}{2}N(W+\varepsilon+\tfrac{1}{2}V) \tag{3}$$

where the periodic boundary condition is assumed. Thus, the problem reduces to the study of the Heisenberg XXZ model in a magnetic field, which is related to the asymmetric six-vertex model [10].

In the quantum inverse scattering method, the local monodromy matrix for the asymmetric six-vertex model is a two by two matrix:

$$L_j = \begin{pmatrix} \dfrac{a_+ + b_+}{2} + \dfrac{a_+ - b_+}{2}\sigma_j^z & c\sigma_j^- \\ c\sigma_j^+ & \dfrac{a_- + b_-}{2} - \dfrac{a_- - b_-}{2}\sigma_j^z \end{pmatrix}. \tag{4}$$

The R matrix satisfying the Yang-Baxter relations

$$R(\lambda,\mu)L_j(\lambda)\otimes L_j(\mu) = L_j(\mu)\otimes L_j(\lambda)R(\lambda,\mu) \tag{5}$$

or, equivalently,

$$R(\lambda,\mu)T_N(\lambda)\otimes T_N(\mu) = T_N(\mu)\otimes T_N(\lambda)R(\lambda,\mu) \tag{6}$$

with

$$T_N(\lambda) = L_N(\lambda)\ldots L_1(\lambda) \tag{7}$$

does exist provided the following constraints are valid:

$$\frac{a_+b_+}{a_-b_-} = \frac{a'_+b'_+}{a'_-b'_-} \tag{8}$$

$$\frac{a_+a_- + b_+b_- - c^2}{a_+b_+ + a_-b_-} = \frac{a'_+a'_- + b'_+b'_- - c'^2}{a'_+b'_+ + a'_-b'_-} \tag{9}$$

in which the prime denotes that the corresponding argument is μ instead of λ. For the Heisenberg XXZ model in a magnetic field, we have introduced the following convenient parametrisation:

$$a_+ : b_+ : a_- : b_- : c = \xi\sin(\lambda+\eta) : \xi^{-1}\sin(\lambda-\eta) : \xi^{-1}\sin(\lambda+\eta) : \xi\sin(\lambda-\eta) : \sin 2\eta \tag{10}$$

with

$$\xi = \sec\alpha\cos(\lambda-\eta+\alpha)\sec(\lambda-\eta). \tag{11}$$

The corresponding R matrix is

$$R(\lambda,\mu) = \begin{pmatrix} a''_+ & 0 & 0 & 0 \\ 0 & c'' & b''_+ & 0 \\ 0 & b''_- & c'' & 0 \\ 0 & 0 & 0 & a''_- \end{pmatrix} \tag{12}$$

with

$$a''_+ : b''_+ : a''_- : b''_- : c''$$
$$= \xi(\lambda)\xi^{-1}(\mu)\sin(\lambda-\mu+2\eta) : \xi(\lambda)\xi(\mu)\sin(\lambda-\mu) :$$
$$\xi^{-1}(\lambda)\xi(\mu)\sin(\lambda-\mu+2\eta) : \xi^{-1}(\lambda)\xi^{-1}(\mu)\sin(\lambda-\mu) : \sin 2\eta. \tag{13}$$

From this we can set up a class of commuting $2^N \times 2^N$ transfer matrices $t_N(\lambda)$,

$$t_N(\lambda) = \mathrm{Tr}[L_N(\lambda)\ldots L_1(\lambda)]. \tag{14}$$

When $\lambda = \eta$ it is quite easy to see that $t_N(\lambda)$ is simply proportional to the operator that shifts all arrows one column to the left. Regarding η and α as constants and differentiating with respect to λ, we can then deduce that

$$H = -J\sin 2\eta \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln t_N(\lambda)\Big|_{\lambda=\eta} + J(\sin 2\eta\tan\alpha + \cos 2\eta)NI_N \tag{15}$$

where I_N is the identity operator and J, V and W are given by

$$J : -\tfrac{1}{2}V : (W+\varepsilon+V) = 1 : \cos 2\eta : 2\, \mathit{sin}\, 2\eta\tan\alpha. \tag{16}$$

Therefore we have concluded that the one-dimensional small-polaron model is related to a special asymmetric six-vertex model described by the Boltzmann weights (10), i.e., it provides a class of commuting row-to-row transfer matrices commuting with the Hamiltonian (1). As usual, the model has higher conserved currents which are involutive with each other. The explicit expression for the first non-trivial conserved current next to the Hamiltonian is given by an expansion of the transfer matrix in powers of λ:

$$\begin{aligned} t_N(\lambda) = t_N(\eta)\Big[1 &- \frac{1}{J\sin^2\eta}[H - \tfrac{1}{2}N(W+\varepsilon)](\lambda-\eta) \\ &+ \frac{1}{2!}\frac{1}{J^2\sin^2 2\eta}[H - \tfrac{1}{2}N(W+\varepsilon)]^2(\lambda-\eta)^2 \\ &- \frac{1}{2!}\frac{1}{J^2\sin^2 2\eta}\left(\frac{(W+\varepsilon+V)^2}{4}\sum_{j=1}^{N}\sigma_j^z - J^2N\right)(\lambda-\eta)^2 \\ &+ \frac{1}{2!}(-\mathrm{i})j(\lambda-\eta)^2 + \ldots\Big] \end{aligned} \tag{17}$$

with

$$\begin{aligned} (-\mathrm{i})j = \frac{1}{J^2\sin^2 2\eta}\sum_{j=1}^{N}\{ & J^2(\sigma_{j+1}^-\sigma_j^z\sigma_{j-1}^+ - \mathrm{HC}) \\ & - \tfrac{1}{2}JV[\sigma_{j+1}^z(\sigma_j^+\sigma_{j-1}^- - \mathrm{HC}) + (\sigma_{j+1}^+\sigma_j^- - \mathrm{HC})\sigma_{j-1}^z]\}. \end{aligned}$$

The others may be obtained in a similar fashion. It is easy to see that our results, as expected, reduce to those previously found by Lüscher [4] in the zero-field limit.

Let us now go over to the fermion operator language. This can be done by using the inverse Jordan-Wigner transformation and introducing a gauge transformation defined by Pu and Zhao:

$$\mathscr{L}_j(\lambda) = V_{j+1}L_j(\lambda)V_j^{-1} \qquad V_j = \begin{pmatrix} \exp\left(\mathrm{i}\tfrac{1}{2}\pi\sum_{l=1}^{j-1} n_l\right) & 0 \\ 0 & \exp\left(-\mathrm{i}\tfrac{1}{2}\pi\sum_{l=1}^{j-1} n_l\right) \end{pmatrix}. \tag{18}$$

Such being the case, the Yang-Baxter relations reduce to the form

$$\mathscr{R}(\lambda,\mu)\mathscr{L}_j(\lambda)\underset{s}{\otimes}\mathscr{L}_j(\mu)=\mathscr{L}_j(\mu)\underset{s}{\otimes}\mathscr{L}_j(\lambda)\mathscr{R}(\lambda,\mu) \tag{19}$$

with

$$\mathscr{L}_j(\lambda)=\begin{pmatrix} b_+-(b_+-\mathrm{i}a_+)n_j & ca_j \\ -\mathrm{i}ca_j^+ & a_--(a_-+\mathrm{i}b_-)n_j \end{pmatrix} \tag{20}$$

and

$$\mathscr{R}(\lambda,\mu)=\begin{pmatrix} a''_+ & 0 & 0 & 0 \\ 0 & c'' & \mathrm{i}b''_+ & 0 \\ 0 & -\mathrm{i}b''_- & c'' & 0 \\ 0 & 0 & 0 & a''_- \end{pmatrix}. \tag{21}$$

Here by $\underset{s}{\otimes}$ we mean the Grassmann direct product [11]

$$[A\underset{s}{\otimes}B]_{\alpha\gamma,\beta\delta}=(-1)^{[P(\alpha)+P(\beta)]P(\gamma)}A_{\alpha\beta}B_{\gamma\delta} \tag{22}$$

with $P(1)=0$, and $P(2)=1$. In this case, the first non-trivial conserved current next to the Hamiltonian (1) expressed in terms of the fermion operator language is

$$\begin{aligned}(-\mathrm{i})\mathscr{J}=\frac{1}{J^2\sin^2 2\eta}\sum_{j=1}^{N}\{&J^2(a_{j+1}a_{j-1}^+-\text{HC})\\ &-JV[n_{j+1}(a_j^+a_{j-1}-\text{HC})+(a_{j+1}^+a_j-\text{HC})n_{j-1}]+JV(a_j^+a_{j-1}-\text{HC})\}\end{aligned} \tag{23}$$

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